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11-2007

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Published in *Finite Fields and Their Applications*, 13, 778-792. doi: [10.1016/j.ffa.2005.11.005](https://doi.org/10.1016/j.ffa.2005.11.005)

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### Recommended Citation

Fitzgerald, Robert W. "Highly Degenerate Quadratic Forms over  $F_2$ ." (Nov 2007).

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# Highly degenerate quadratic forms over $F_2$

Robert W. Fitzgerald

## Abstract

Let  $K$  be a finite extension of  $F_2$ . We consider quadratic forms written as the trace of  $xR(x)$ , where  $R(x)$  is a linearized polynomial. We determine the  $K$  and  $R(x)$  where the form has a radical of codimension 2. This is applied to constructing maximal Artin-Schreier curves.

Set  $F = F_2$  and let  $K = F_{2^k}$  be an extension of degree  $k$ . Let

$$R_{\bar{\epsilon}}(x) = \sum_{i=0}^m \epsilon_i x^{2^i},$$

with each  $\epsilon_i \in K$  and either  $k = 2m$  or  $k = 2m + 1$ . We consider the quadratic forms  $Q_{\bar{\epsilon}}^K : K \rightarrow F$  given by  $Q_{\bar{\epsilon}}^K(x) = \text{tr}_{K/F}(xR_{\bar{\epsilon}}(x))$ .

These trace forms have appeared in a variety of contexts. They have been used to compute weight enumerators of certain binary codes [1], [2], to construct curves with many rational points and the associated trace codes [9], as part of an authentication scheme [3], and to construct certain binary sequences in [6] and [5].

In each of these applications one wants the number of solutions (in  $K$ ) to  $Q_{\bar{\epsilon}}^K(x) = 0$ , denoted by  $N(Q_{\bar{\epsilon}}^K)$ . This is easily worked out (see [8], 6.26, 6.32) in terms of the standard classification of quadratic forms:

$$N(Q_{\bar{\epsilon}}^K) = \frac{1}{2}(2^k + \Lambda(Q_{\bar{\epsilon}}^K)\sqrt{2^{k+w}}). \quad (1)$$

where  $w$  is the dimension of the radical,  $v = (k - w)/2$  and

$$\Lambda(Q_{\bar{\epsilon}}^K) = \begin{cases} 0, & \text{if } Q_{\bar{\epsilon}}^K \simeq z^2 + \sum_{i=1}^v x_i y_i \\ 1, & \text{if } Q_{\bar{\epsilon}}^K \simeq \sum_{i=1}^v x_i y_i \\ -1, & \text{if } Q_{\bar{\epsilon}}^K \simeq x_1^2 + y_1^2 + \sum_{i=1}^v x_i y_i. \end{cases}$$

However, there is no simple way to determine the dimension of the radical or the invariant  $\Lambda$ . The one general result is due to Klapper [7] which only covers the case when  $R$  consists of a single term. In roughly half the applications ([1], [2], [9]) one wants highly degenerate forms, which give large  $N(Q_{\bar{\varepsilon}}^K)$  when  $\Lambda = 1$ . In a previous paper [4] we considered only those  $R$  with all coefficients  $\varepsilon_i \in F_2$  but allowed  $F$  to be any finite field of characteristic 2. Here we restrict to  $F = F_2$ , the case of most applications, and allow arbitrary coefficients  $\varepsilon_i$ . Our main result is to determine all such  $R_{\bar{\varepsilon}}$ , and all extensions  $K$ , such that the radical of  $Q_{\bar{\varepsilon}}^K$  has codimension at most 2. We compute the invariant  $\Lambda$  in each case. We apply the result to a classification of maximal Artin-Schreier curves  $y^2 = xR_{\bar{\varepsilon}}(x)$ .

The two situations,  $\varepsilon_i \in F_2$  (treated in [4]) and  $F = F_2$  (treated here), look similar but are in fact quite different. For instance, when  $K = F_{2^9}$ , there are exactly two quadratic forms of codimension 2 radical with  $\varepsilon_i \in F_2$  but over 22 million with arbitrary  $\varepsilon_i$ . The classification here is then not a list of possible  $R_{\bar{\varepsilon}}$  but formulas showing how an arbitrary  $\varepsilon_0, \varepsilon_1$  determine the other  $\varepsilon_i$ . We also give formulas for the number of forms of codimension 2 radical of each invariant  $\Lambda$ .

## 1 Determining the coefficients

We recall the basic result from [4].

**Lemma 1.1.** (1) *The radical of  $Q_{\bar{\varepsilon}}^K$  does not depend on  $\varepsilon_0$ , that is, if  $\varepsilon_i = \varepsilon'_i$  for  $i \geq 1$  then*

$$\text{rad}(Q_{\bar{\varepsilon}}^K) = \text{rad}(Q_{\bar{\varepsilon}'}^K).$$

(2) *We have  $\dim \text{rad}(Q_{\bar{\varepsilon}}^K) = k - 2$  iff there exist independent (over  $F$ )  $a, b \in K$  such that*

$$\varepsilon_i = a^{2^i}b + ab^{2^i}, \tag{2}$$

*for  $1 \leq i \leq m$ , except when  $k = 2m$  in which case  $\varepsilon_m \equiv a^{2^m}b \pmod{F_{2^m}}$ . Moreover,  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$  if  $\varepsilon_0 = ab$ ,  $\Lambda(Q_{\bar{\varepsilon}}^K) = -1$  if  $\varepsilon_0 = a^2 + ab + b^2$  and  $\Lambda(Q_{\bar{\varepsilon}}^K) = 0$  in all other cases.*

If Equation 2 holds then  $\varepsilon_1 = ab(a + b)$  so that  $\varepsilon_1 = 0$  implies  $a = 0$ ,  $b = 0$  or  $a = b$ . This contradicts the independence of  $a, b$ . Hence if  $Q_{\bar{\varepsilon}}^K$  has

a codimension 2 radical then  $\varepsilon_1 \neq 0$ . We will use this fact constantly, and without further mention, throughout.

Set  $h = \lfloor (k-1)/2 \rfloor$ . Then Equation 2 holds for  $1 \leq i \leq h$  (that is, we have excluded the exceptional case  $i = m$  when  $k = 2m$ ).

**Lemma 1.2.**  $Q_\varepsilon^K$  has a codimension 2 radical iff there exists a  $v \in K^*$  such that each of the following holds:

1.  $\varepsilon_1 \neq 0$  and  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$ .
2. For all  $2 \leq i \leq h$ ,

$$v^{2^{i-1}} \varepsilon_i = \varepsilon_1^{2^{i-1}} \varepsilon_{i-1} + \varepsilon_1 v^{2^i-1}.$$

3. If  $k = 2m$  then  $\varepsilon_m \equiv a^{2^m} b \pmod{F_{2^m}}$ , where  $a, b$  are the roots of (1).

**Proof:** First suppose  $Q_\varepsilon^K$  has a codimension 2 radical. Let  $a, b \in K$  be the elements giving Lemma 1.1 (2). Set  $u = a + b$  and  $v = ab$ . Then  $\varepsilon_1 = a^b + ab^2 = uv$ . Hence

$$y^2 + \frac{\varepsilon_1}{v}y + v = (y+a)(y+b)$$

splits in  $K$ .

From [4] p. 173,

$$v \sum_{j=0}^{i-1} u^{2^i-2^{j+1}+1} v^{2^j-1} = \varepsilon_i.$$

Replacing  $u$  by  $\varepsilon_1/v$ , and multiplying by  $v^{2^i-2}$  gives

$$\sum_{j=0}^{i-1} \varepsilon_1^{2^i-2^{j+1}+1} v^{3(2^j-1)} = \varepsilon_i v^{2^i-2}.$$

Hence

$$\begin{aligned} \varepsilon_1^{2^{i-1}} \sum_{j=0}^{i-2} \varepsilon_1^{2^{i-1}-2^{j+1}+1} v^{3(2^j-1)} + \varepsilon_1 v^{3(2^{i-1}-1)} &= \varepsilon_i v^{2^i-2} \\ \varepsilon_1^{2^{i-1}} \varepsilon_{i-1} v^{2^{i-1}-2} + \varepsilon_1 v^{3(2^{i-1}-1)} &= \varepsilon_i v^{2^i-2} \\ \varepsilon_1^{2^{i-1}} \varepsilon_{i-1} + \varepsilon_1 v^{2^i-1} &= \varepsilon_i v^{2^{i-1}}, \end{aligned}$$

which gives (2). And (3) follows Lemma 1.1.

Now suppose (1), (2) and (3) hold. Let  $a, b \in K$  be the roots of (1). Note that  $a$  and  $b$  are independent over  $F$  as  $v \neq 0$  shows  $a \neq 0$  and  $b \neq 0$ , while  $\varepsilon_1 \neq 0$  shows  $a \neq b$ . Note that  $v = ab$  and  $\varepsilon_1/v = a + b$ . So  $\varepsilon_1 = a^2b + ab^2$ . We show by induction that Lemma 1.1 (2) holds for all  $2 \leq i \leq h$ . We have by (2) and induction that

$$\begin{aligned} (ab)^{2^i} \varepsilon_{i+1} &= [ab(a+b)]^{2^i} (a^{2^i}b + ab^{2^i}) + ab(a+b)(ab)^{2^{i+1}-1} \\ \varepsilon_{i+1} &= (a^{2^i} + b^{2^i})(a^{2^i}b + ab^{2^i}) + (ab)^{2^i}(a+b) \\ &= a^{2^{i+1}}b + ab^{2^{i+1}}. \end{aligned}$$

This, with (3), shows  $Q_{\varepsilon}^K$  has a codimension 2 radical.  $\square$

We want a formula for  $\varepsilon_i$  that does not depend on  $v$ , only on the initial coefficients  $\varepsilon_1$  and  $\varepsilon_2$ . Now

$$\begin{aligned} v^3 &= \frac{\varepsilon_2}{\varepsilon_1} v^2 + \varepsilon_1^2 \\ v^4 &= \frac{\varepsilon_2}{\varepsilon_1} \varepsilon v^3 + \varepsilon_1^2 v = \frac{\varepsilon_2^2}{\varepsilon_1^2} v^2 + \varepsilon_1^2 v + \varepsilon_1 \varepsilon_2. \end{aligned}$$

Hence for each  $i \geq 0$  we can write

$$v^{2^i} = A_i v^2 + B_i v + C_i,$$

where  $A_i, B_i, C_i \in F(\varepsilon_1, \varepsilon_2)$ .

**Lemma 1.3.** *Suppose  $v^2 \varepsilon_2 = \varepsilon_1^3 + v^3 \varepsilon_1$ .*

1. *We have  $A_1 = 1, B_1 = 0, C_1 = 0$  and for each  $i \geq 1$*

$$A_{i+1} = \frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 + B_i^2 \quad B_{i+1} = \varepsilon_1^2 A_i^2 \quad C_{i+1} = \varepsilon_1 \varepsilon_2 A_i^2 + C_i^2.$$

2.  *$C_i^2 = \varepsilon_1^2 A_i B_i$  for  $i \geq 1$ .*

3.  *$\varepsilon_2^2 A_i^2 B_i + \varepsilon_1^4 A_i^3 + \varepsilon_1^2 B_i^3 = \varepsilon_1^{2^{i+1}}$  for  $i \geq 1$ .*

4.  *$(\varepsilon_2 A_i + \varepsilon_1 B_i) C_i + \varepsilon_1^3 A_i^2 = \varepsilon_1^{2^i} (\varepsilon_2 A_{i-1} \varepsilon + \varepsilon_1 B_{i-1})$  for all  $i \geq 1$ .*

**Proof:** (1) We have

$$\begin{aligned} v^{2^{i+1}} &= A_i^2 v^4 + B_i^2 v^2 + C_i^2 \\ &= \left(\frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 + B_i^2\right) v^2 + \varepsilon_1^2 A_i^2 v + (\varepsilon_1 \varepsilon_2 A_i^2 + C_i^2). \end{aligned}$$

(2) This is true for  $i = 1$ . And

$$\begin{aligned} \varepsilon_1^2 A_{i+1} B_{i+1} &= (\varepsilon_2^2 A_i^2 + \varepsilon_1^2 B_i^2) \varepsilon_1^2 A_i^2 \\ &= \varepsilon_1^2 \varepsilon_2^2 A_i^4 + \varepsilon_1^4 A_i^2 B_i^2 \\ &= \varepsilon_1^2 \varepsilon_2^2 A_i^4 + C_i^4 \quad (\text{by induction}) \\ &= C_{i+1}^2, \end{aligned}$$

by (1).

(3) This is true for  $i = 1$ . And

$$\begin{aligned} \varepsilon_2^2 A_{i+1}^2 B_{i+1} &+ \varepsilon_1^4 A_{i+1}^3 + \varepsilon_1^2 B_{i+1}^3 \\ &= \left(\frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 + B_i^2\right)^2 \varepsilon_1^2 \varepsilon_2^2 A_i^2 + \varepsilon_1^4 \left(\frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 \varepsilon^2 + B_i^2\right)^3 + \varepsilon_1^2 (\varepsilon_1^2 A_i^2)^3 \\ &= \varepsilon_2^4 A_i^4 B_i^2 + \varepsilon_1^4 B_i^6 + \varepsilon_1^8 A_i^6 \\ &= (\varepsilon_2^2 A_i^2 B_i + \varepsilon_1^4 A_i^3 + \varepsilon_1^2 B_i^3)^2 = (\varepsilon_1^{2^{i+1}})^2 = \varepsilon_1^{2^{i+2}}, \end{aligned}$$

using induction.

(4) This is true for  $i = 1$ . And

$$\begin{aligned} (\varepsilon_2 A_{i+1} &+ \varepsilon_1 B_{i+1}) C_{i+1} + \varepsilon_1^3 A_{i+1}^2 \\ &= (\varepsilon_2 \left[\frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 + B_i^2\right] + \varepsilon_1^3 A_i^2) (\varepsilon_1 \varepsilon_2 A_i^2 + C_i^2) + \varepsilon_1^3 \left(\frac{\varepsilon_2^2}{\varepsilon_1^2} A_i^2 + B_i^2\right)^2 \\ &= \left(\frac{\varepsilon_2^3}{\varepsilon_1^2} A_i^2 C_i^2 + \varepsilon_2 B_i^2 C_i^2\right) + \varepsilon_1^4 \varepsilon_2 A_i^4 + (\varepsilon_1 \varepsilon_2^2 A_i^2 B_i^2 + \varepsilon_1^2 A_i^2 C_i^2 + \varepsilon_1^3 B_i^4) \\ &= \frac{\varepsilon_2}{\varepsilon_1^2} C_i^2 (\varepsilon_2^2 A_i^2 + \varepsilon_1^2 B_i^2) + \varepsilon_1^4 \varepsilon_2 A_i^4 + \varepsilon_1 B_i (\varepsilon_2^2 A_i^2 B_i + \varepsilon_1^4 A_i^3 + \varepsilon_1^2 B_i^3) \\ &= \frac{\varepsilon_2}{\varepsilon_1^2} (\varepsilon_1^3 A_i^2 + \varepsilon_1^{2^i} (\varepsilon_2 A_{i-1} + \varepsilon_1 B_{i-1}))^2 + \varepsilon_1^4 \varepsilon_2 A_i^4 + \varepsilon_1 B_i \varepsilon_1^{2^{i+1}} \\ &= \frac{\varepsilon_2}{\varepsilon_1^2} \varepsilon_1^{2^{i+1}} (\varepsilon_2^2 A_{i-1}^2 + \varepsilon_1^2 B_{i-1}^2) + \varepsilon_1 B_i \varepsilon_1^{2^{i+1}} \\ &= \varepsilon_1^{2^{i+1}} (\varepsilon_2 A_i + \varepsilon_1 B_i). \end{aligned}$$

Here the third line uses (2) while the fourth line uses induction and (3).  $\square$

**Proposition 1.4.** *Suppose  $v^2\varepsilon_2 = \varepsilon_1v^3 + \varepsilon_1^3$ . Then Lemma 1.2 (2) holds iff  $\varepsilon_{i+1} = \varepsilon_2A_i + \varepsilon_1B_i = \varepsilon_1\sqrt{A_{i+1}}$  for all  $i \geq 1$ .*

**Proof:** Suppose Lemma 1.2 (2) holds. The second equation is true for  $i = 1$ . For  $i > 1$ , Lemma 1.3 (1) gives :

$$\begin{aligned}\varepsilon_{i+1} &= \frac{\varepsilon_1v^{2^{i+1}-1} + \varepsilon_1^{2^i}\varepsilon_i}{v^{2^i}} \\ &= \frac{\frac{\varepsilon_1}{v}(A_{i+1}v^2 + B_{i+1}v + C_{i+1}) + \varepsilon_1^{2^i}\varepsilon_i}{A_iv^2 + B_iv + C_i}.\end{aligned}$$

Now

$$\frac{1}{v} = \frac{1}{\varepsilon_1^2}v^2 + \frac{\varepsilon_2}{\varepsilon_1^3}v.$$

Then

$$\begin{aligned}\varepsilon_{i+1} &= \frac{\frac{1}{\varepsilon_1}C_{i+1}v^2 + (\frac{\varepsilon_2}{\varepsilon_1^2}C_{i+1} + \varepsilon_1A_{i+1})v + \varepsilon_1B_{i+1} + \varepsilon_1^{2^i}\varepsilon_i}{A_iv^2 + B_iv + C_i} \\ &= \frac{(\varepsilon_2A_i^2 + \frac{1}{\varepsilon_1}C_i^2)v^2 + (\frac{\varepsilon_2}{\varepsilon_1^2}C_i^2 + \varepsilon_1B_i^2)v + \varepsilon_1^3A_i^2 + \varepsilon_1^{2^i}\varepsilon_i}{A_iv^2 + B_iv + C_i}.\end{aligned}$$

By Lemma 1.3 (2)

$$\begin{aligned}(\varepsilon_2A_i + \varepsilon_1B_i)A_i &= \varepsilon_2A_i^2 + \varepsilon_1A_iB_i = \varepsilon_2A_i^2 + \frac{1}{\varepsilon_1}C_i^2 \\ (\varepsilon_2A_i + \varepsilon_1B_i)B_i &= \varepsilon_2A_iB_i + \varepsilon_1B_i^2 = \frac{\varepsilon_2}{\varepsilon_1^2}C_i^2 + \varepsilon_1B_i^2.\end{aligned}$$

Lastly, from Lemma 1.3 (4)

$$\begin{aligned}(\varepsilon_2A_i + \varepsilon_1B_i)C_i &= \varepsilon_1^3A_i^2 + \varepsilon_1^{2^i}(\varepsilon_2A_{i-1} + \varepsilon_1B_{i-1}) \\ &= \varepsilon_1^3A_i^2 + \varepsilon_1^{2^i}\varepsilon_i,\end{aligned}$$

using induction. Hence  $\varepsilon_{i+1} = \varepsilon_2A_i + \varepsilon_1B_i$ .

For the converse, the steps may be reversed.  $\square$

**Theorem 1.5.**  $Q_{\bar{\varepsilon}}^K$  has codimension 2 radical iff there exists  $v \in K^*$  such that each of the following holds

1.  $\varepsilon_1 \neq 0$  and  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$ ,

2.  $v^2\varepsilon_2 = \varepsilon_1^3 + \varepsilon_1v^3,$

3. for  $i \geq 2$  we have

$$\varepsilon_{i+1} = \frac{\varepsilon_2}{\varepsilon_1^2}\varepsilon_i^2 + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4,$$

4. if  $k = 2m$  then  $\varepsilon_m \equiv a^{2^m}b \pmod{F_{2^m}}$ , where  $a, b$  are the roots of (1).

Moreover, in this case,

$$\Lambda(Q_{\varepsilon}^K) = \begin{cases} 1, & \text{if } \epsilon_0 = v \\ -1, & \text{if } \epsilon_0 = v + (\varepsilon_1/v)^2 \\ 0, & \text{otherwise.} \end{cases}$$

**Proof:** We need to show that Lemma 1.1 (2) is equivalent to the statement (3) here, given (1) and (2). We first check that Lemma 1.1 implies (3).

$$\begin{aligned} \varepsilon_{i+1} &= \varepsilon_2A_i + \varepsilon_1B_i && \text{by Proposition 1.4} \\ &= \varepsilon_2(\sqrt{A_i})^2 + \varepsilon_1^3(\sqrt{A_{i-1}})^4 && \text{by Lemma 1.3 (1)} \\ &= \frac{\varepsilon_2}{\varepsilon_1^2}\varepsilon_i^2 + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4 && \text{by Proposition 1.4.} \end{aligned}$$

Next we check that (3) implies Lemma 1.1 (2). It is enough to show  $\varepsilon_{i+1} = \varepsilon_2A_i + \varepsilon_1B_i$ , by Proposition 1.4. We use induction. The case  $i = 1$  is clear.

$$\begin{aligned} \varepsilon_{i+1} &= \frac{\varepsilon_2}{\varepsilon_1^2}\varepsilon_i^2 + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4 && \text{by (3)} \\ &= \frac{\varepsilon_2}{\varepsilon_1^2}(\varepsilon_2A_{i-1} + \varepsilon_1B_{i-1})^2 + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4 && \text{by induction} \\ &= \frac{\varepsilon_2^3}{\varepsilon_1^2}A_{i-1}^2 + \varepsilon_2B_{i-1}^2 + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4 \\ &= \varepsilon_2 \left( \frac{\varepsilon_2^2}{\varepsilon_1^2}A_{i-1}^2 + B_{i-1}^2 \right) + \frac{1}{\varepsilon_1}\varepsilon_{i-1}^4 \\ &= \varepsilon_2A_i + \frac{1}{\varepsilon_1}(\varepsilon_1\sqrt{A_{i-1}})^4 && \text{by Lemma 1.3 (1) and induction} \\ &= \varepsilon_2A_i + \varepsilon_1B_i, \end{aligned}$$



using Lemma 1.3 (1) again.

Lastly, we check the invariants. As  $a, b$  are roots of  $y^2 + (\varepsilon_1/v)y + v$ , we have  $v = ab$  and  $\varepsilon_1/v = a + b$ . Now, by Lemma 1.1,  $\Lambda(Q_{\varepsilon}^K) = +1$  iff  $\varepsilon_0 = ab$ , which is  $v$ . And  $\Lambda(Q_{\varepsilon}^K) = -1$  iff  $\varepsilon_0 = a^2 + ab + b^2 = v + (\varepsilon_1/v)^2$ .  $\square$

**Proposition 1.6.** *Equation (3) of Theorem 1.5 is equivalent to:*

$$\varepsilon_i = \frac{\varepsilon_2^{s_i}}{\varepsilon_1^{\ell_i}} \sum_{j \in \Delta_i} \varepsilon_1^{5(t_i-j)} \varepsilon_2^{3j},$$

where  $\ell_i = 2^{i-1} - 2$ ,

$$s_i = \begin{cases} 0, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even,} \end{cases} \quad t_i = \begin{cases} (2^{i-1} - 1)/3, & \text{if } i \text{ is odd} \\ (2^{i-1} - 2)/3, & \text{if } i \text{ is even,} \end{cases}$$

$\Delta_3 = \{0, 1\}$ ,  $\Delta_4 = \{0, 1, 2\}$  and for  $i \geq 5$

$$\Delta_i = (A_i + \{0, 1, 2\}) \cup (2A_{i-1} + \{4, 5\})$$

where

$$A_i = \{0\} \cup \{2^{n_1} + 2^{n_2} + \cdots + 2^{n_r} : n_j - n_{j-1} \geq 2, n_1 \leq i - 3, n_r \geq 3\}.$$

**Proof:** Assume first that Theorem 1.5 (3) holds. The formulas for  $\varepsilon_3$  and  $\varepsilon_4$  can be checked directly. We use induction. Suppose  $i$  is odd (the case of  $i$  even is similar). Then

$$\begin{aligned} s_{i-1} &= 0 & s_i &= 1 & s_{i+1} &= 0 \\ 2\ell_{i-1} &= \ell_i - 2 & 2\ell_i &= \ell_{i+1} - 2 \\ 2t_{i-1} &= t_i + 1 & 2t_i &= t_{i+1}. \end{aligned}$$

We have

$$\begin{aligned} \varepsilon_{i+1} &= \frac{\varepsilon_2}{\varepsilon_1^2} \varepsilon_i^2 + \frac{1}{\varepsilon_1} \varepsilon_{i-1}^4 \\ &= \frac{\varepsilon_2}{\varepsilon_1^2} \left[ \frac{1}{\varepsilon_1^{\ell_i}} \sum_{j \in \Delta_i} \varepsilon_i^{5(t_i-j)} \varepsilon_2^{3j} \right]^2 + \frac{1}{\varepsilon_1} \left[ \frac{\varepsilon_2}{\varepsilon_1^{\ell_{i-1}}} \sum_{j \in \Delta_{i-1}} \varepsilon_1^{5(t_{i-1}-j)} \varepsilon_2^{3j} \right]^4 \\ &= \frac{\varepsilon_2}{\varepsilon_1^{2\ell_i+2}} \sum_{j \in \Delta_i} \varepsilon_1^{5(2t_i-2j)} \varepsilon_2^{3(2j)} + \frac{\varepsilon_2^4}{\varepsilon_1^{2\ell_i-3}} \sum_{j \in \Delta_{i-1}} \varepsilon_1^{5(2t_i-4j-2)} \varepsilon_2^{3(4j)} \\ &= \frac{\varepsilon_2}{\varepsilon_1^{\ell_{i+1}}} \left[ \sum_{j \in \Delta_i} \varepsilon_1^{5(t_{i+1}-2j)} \varepsilon_2^{3(2j)} + \sum_{j \in \Delta_{i-1}} \varepsilon_1^{5(t_{i+1}-4j-1)} \varepsilon_2^{3(4j+1)} \right]. \end{aligned}$$

Note that there are no terms in common to the two sums. Set

$$\Delta_{i+1} = 2\Delta_i \cup (4\Delta_{i-1} + 1).$$

Then we have

$$\varepsilon_{i+1} = \frac{\varepsilon_2^{s_{i+1}}}{\varepsilon_1^{\ell_{i+1}}} \sum_{j \in \Delta_{i+1}} \varepsilon_1^{5(t_{i+1}-j)} \varepsilon_2^{3j}.$$

Hence we need only check that  $\Delta_{i+1} = (A_{i+1} + \{0, 1, 2\}) \cup (2A_{i-1} + \{4, 5\})$ . We do this by induction on  $i$ .

**Claim**  $A_{i+1} = 2A_i \cup (4A_{i-1} + 8)$ . The inclusion  $\supset$  is easy to check. Suppose  $\alpha = 2^{n_1} + \dots + 2^{n_r} \in A_{i+1}$ . If  $n_r \geq 4$  then

$$\alpha = 2(2^{n_1-1} + \dots + 2^{n_r-1}) \in 2A_i.$$

If  $n_r = 3$  then  $n_{r-1} \geq 5$  and

$$\alpha = 4(2^{n_1-2} + \dots + 2^{n_{r-1}-2}) + 8 \in 4A_{i-1} + 8,$$

proving the **Claim**.

We have  $\Delta_{i+1} = 2\Delta_i \cup (4\Delta_{i-1} + 1)$  so by induction

$$\begin{aligned} \Delta_{i+1} &= (2A_i + \{0, 2, 4\}) \cup (4A_{i-1} + \{8, 10\}) \\ &\quad \cup (4A_{i-1} + \{1, 5, 9\}) \cup (8A_{i-2} + \{17, 21\}). \end{aligned}$$

Now by the **Claim**

$$\begin{aligned} A_{i+1} &= 2A_i \cup (4A_{i-1} + 8) \\ A_{i+1} + 2 &= (2A_i + 2) \cup (4A_{i-1} + 10) \\ A_{i+1} + 1 &= (2A_i + 1) \cup (4A_{i-1} + 9) \\ &= 2[2A_{i-1} \cup (4A_{i-2} + 8)] + 1 \cup (4A_{i-1} + 9) \\ &= (4A_{i-1} + 1) \cup (8A_{i-2} + 17) \cup (4A_{i-1} + 9) \\ 2A_i + 5 &= 2[2A_{i-1} \cup (4A_{i-2} + 8)] + 5 \\ &= (4A_{i-1} + 5) \cup (8A_{i-2} + 21). \end{aligned}$$

Thus  $\Delta_{i+1} = (A_{i+1} + \{0, 1, 2\}) \cup (2A_i + \{4, 5\})$ .

The converse follows from a simple, but tedious, substitution.  $\square$

Here are the first few  $\varepsilon_i$ :

$$\begin{aligned}\varepsilon_3 &= \frac{1}{\varepsilon_1^2}(\varepsilon_2^3 + \varepsilon_1^5) \\ \varepsilon_4 &= \frac{\varepsilon_2}{\varepsilon_1^6}(\varepsilon_2^6 + \varepsilon_1^5 \varepsilon_2^3 + \varepsilon_1^{10}) \\ \varepsilon_5 &= \frac{1}{\varepsilon_1^{14}}(\varepsilon_2^{15} + \varepsilon_1^5 \varepsilon_2^{12} + \varepsilon_1^{10} \varepsilon_2^9 + \varepsilon_1^{20} \varepsilon_2^3 + \varepsilon_1^{25}) \\ \varepsilon_6 &= \frac{\varepsilon_2}{\varepsilon_1^{30}}(\varepsilon_2^{30} + \varepsilon_1^5 \varepsilon_2^{27} + \varepsilon_1^{10} \varepsilon_2^{24} + \varepsilon_1^{20} \varepsilon_2^{18} + \varepsilon_1^{25} \varepsilon_2^{15} + \varepsilon_1^{40} \varepsilon_2^6 + \varepsilon_1^{45} \varepsilon_2^3 + \varepsilon_1^{50}).\end{aligned}$$

The next two  $\Delta_i$  are:

$$\begin{aligned}\Delta_7 &= \{0, 1, 3, 4, 5, 11, 12, 13, 16, 17, 19, 20, 21\} \\ \Delta_8 &= \{0, 1, 2, 5, 6, 8, 9, 10, 21, 22, 24, 25, 26, 32, 33, 34, 37, 38, 40, 41, 42\}.\end{aligned}$$

## 2 Construction and Examples

**Construction:** Choose any  $\varepsilon_0 \in K$  and  $\varepsilon_1 \in K^*$ . Find all  $v \in K^*$  such that  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$ . Set  $\varepsilon_2 = (\varepsilon_1 v^3 + \varepsilon_1^3)/v^2$ . Let  $\varepsilon_i$ , for  $3 \leq i \leq \lfloor (k-1)/2 \rfloor$ , be given by Corollary 1.6. If  $k = 2m$  then set  $\varepsilon_m = a^{2^m}b$ , where  $a, b$  are the roots of  $y^2 + (\varepsilon_1/v)y + v$ . Take

$$R_{\bar{\varepsilon}} = \sum_{j=0}^m \varepsilon_j x^{2^j}.$$

**Corollary 2.1.** *The construction gives all  $R_{\bar{\varepsilon}}$  such that  $Q_{\bar{\varepsilon}}^K$  has a codimension 2 radical.*

**Proof:** This is a re-statement of Theorem 1.5. □

We wish to count the number of such  $R_{\bar{\varepsilon}}$ .

**Lemma 2.2.** *The number  $S$  of  $v \in K^*$  such that  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$  is*

$$S = \begin{cases} \frac{1}{2}(2^k - 2), & \text{if } k \text{ is odd} \\ \frac{1}{2}(2^k - (-1)^m 2^{m+1} - 2), & \text{if } k = 2m \text{ and } \varepsilon_1 \in K^{*3} \\ \frac{1}{2}(2^k + (-1)^m 2^m - 2), & \text{if } k = 2m \text{ and } \varepsilon_1 \notin K^{*3}. \end{cases}$$

**Proof:** Let  $q : K \rightarrow F$  be  $q(x) = \text{tr}_{K/F}(\epsilon_1^{-2}x^3)$ . We first check that  $S = N(q) - 1$ , where  $N(q)$  denotes the number of zeros of  $q$  in  $K$ . Now

$$\frac{v^2}{\epsilon_1^2}(y^2 + \frac{\epsilon_1}{v}y + v) = \left(\frac{vy}{\epsilon_1}\right)^2 + \left(\frac{vy}{\epsilon_1}\right) + \frac{v^3}{\epsilon_1^2}.$$

Hence  $y^2 + (\epsilon_1/v)y + v$  splits in  $K$  iff  $s^2 + s + (v^3/\epsilon_1^2)$  splits in  $K$  iff we have  $\text{tr}_{K/F}(v^3/\epsilon_1^2) = 0$  iff  $q(v) = 0$ .

Now we use Equation 1 and Klapper's classification [7] which says, for this case,

1.  $\text{rad } q \neq 0$  iff  $\epsilon_1^{-2} \in K^{*3}$  iff  $\epsilon_1 \in K^{*3}$ , in which case  $\dim \text{rad } q = 2$ .
2.  $q(\text{rad } q) \neq 0$  iff  $k$  is odd.
3. If  $\epsilon_1 \in K^{*3}$  and  $k = 2m$  then

$$\Lambda(q) = \begin{cases} 1, & \text{if } m \text{ is odd} \\ -1, & \text{if } m \text{ is even.} \end{cases}$$

If  $\epsilon_1 \notin K^{*3}$  and  $k = 2m$  then

$$\Lambda(q) = \begin{cases} 1, & \text{if } m \text{ is even} \\ -1, & \text{if } m \text{ is odd.} \end{cases}$$

The result now follows. □

**Theorem 2.3.** *The number of  $R_{\bar{\epsilon}}$  over  $K$  with  $Q_{R_{\bar{\epsilon}}}^K$  having a codimension 2 radical is*

$$\frac{1}{6}q(q-1)(q-2) = \binom{q}{3}.$$

*For a fixed  $\epsilon_1, \epsilon_2$ , three have invariant +1, one has invariant -1 and the rest have invariant 0.*

**Proof:** First note that there are  $q$  choices for  $\epsilon_0$ ,  $q-1$  choices for the non-zero  $\epsilon_1$ . We **Claim** that there are  $S/3$  choices for  $\epsilon_2$ . Fix  $\epsilon_1 \in K^*$ . For each  $v \in K^*$  such that  $y^2 + (\epsilon_1/v)y + v$  has roots in  $K$ , say  $a(v)$  and  $b(v)$ , we get an  $\epsilon_2(v)$  via  $\epsilon_2(v) = a(v)^4b(v) + a(v)b(v)^4$ .

Let  $v_1$ ,  $a(v_1)$ ,  $b(v_1)$  and  $\varepsilon_2(v_1)$  be one choice. Note  $a(v_1)b(v_1) = v_1$  and  $a(v_1) + b(v_1) = \varepsilon_1/v$ . Set  $v_2 = a(v_1)(a(v_1) + b(v_1))$ . Then

$$\frac{\varepsilon_1}{v_2} = \frac{1}{a(v_1)} \frac{\varepsilon_1}{a(v_1) + b(v_1)} = \frac{v_1}{a(v_1)} = b(v_1).$$

Hence  $y^2 + (\varepsilon_1/v_2)y + v_2$  has roots  $a(v_2) = a(v_1)$  and  $b(v_2) = a(v_1) + b(v_1)$ . So

$$\begin{aligned} \varepsilon_2(v_2) &= a(v_1)^4(a(v_1) + b(v_1)) + a(v_1)(a(v_1) + b(v_1))^4 \\ &= a(v_1)^4b(v_1) + a(v_1)b(v_1)^4 = \varepsilon_2(v_1). \end{aligned}$$

Similarly, if  $v_3 = (a(v_1) + b(v_1))b(v_1)$  then  $\varepsilon_2(v_3) = \varepsilon_2(v_1)$  also. Hence there are at least three  $v$ 's giving the same  $\varepsilon_2$ . And we have  $\varepsilon_1v^3 + \varepsilon_1^3 = \varepsilon_2v^2$ , by Lemma 1.2 (2), so there are exactly three  $v$ 's giving the same  $\varepsilon_2$ . Thus the number of  $\varepsilon_2$  is  $S/3$ , proving the **Claim**. The other  $\varepsilon_i$  are determined by Theorem 1.5.

This shows the number of  $R_{\bar{\varepsilon}}$  is  $q(q-1)S/3$ . When  $k$  is odd, this is the desired formula, by Lemma 2.2. Now say  $k = 2m$  is even. Then  $|K^{*3}| = (q-1)/3$ . So, again using Lemma 2.2, the number of  $R_{\bar{\varepsilon}}$  is

$$\begin{aligned} q \frac{q-1}{3} \cdot \frac{1}{3} (2^{k-1} - (-1)^m 2^m - 1) &+ q \frac{2(q-1)}{3} \cdot \frac{1}{3} (2^{k-1} + (-1)^m 2^{m-1} - 1) \\ &= \frac{q}{9} (2^k - 1) (2^k + 2^{k-1} - 3) \\ &= \frac{q}{3} (2^k - 1) (2^{k-1} - 1) \\ &= \frac{1}{6} q (q-1) (q-2). \end{aligned}$$

Lastly, fix  $\varepsilon_1$  and  $\varepsilon_2$ . The invariant  $\Lambda$  is  $+1$  iff  $\varepsilon_0 = v$ , by Theorem 1.5. We have previously shown there are three  $v$ 's giving the same  $\varepsilon_2$ , so there are three  $\varepsilon_0$ 's with  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$ . Again by Theorem 1.5,  $\Lambda(Q_{\bar{\varepsilon}}^K) = -1$  iff  $\varepsilon_0 = v + (\varepsilon_1/v)^2 = a(v)^2 + a(v)b(v) + b(v)^2$ . The three choices for  $(a(v), b(v))$  are  $(a, b)$ ,  $(a, a+b)$ ,  $(a+b, b)$ . In each case,  $a(v)^2 + a(v)b(v) + b(v)^2$  is  $a^2 + ab + b^2$ . Thus there is only one  $\varepsilon_0$  giving an invariant of  $-1$ .  $\square$

**Example 2.4.** Let  $K = F_{2^6}$  with primitive element  $\alpha$ , a root of  $x^6 + x + 1$ . Suppose  $\varepsilon_1 = \alpha$ , a non-cube. A simple computer search will show there are 9

possible  $\varepsilon_2$ 's, in agreement with by Theorem 2.3. Then  $\varepsilon_3$  can be computed, from the exceptional case of the Construction and

$$R_{\bar{\varepsilon}} = \varepsilon_0 x + \varepsilon_1 x^2 + \varepsilon_2 x^4 + \varepsilon_3 x^8.$$

The values are (recall that  $\varepsilon_3$  is only defined modulo  $F_8$ ):

$\varepsilon_2$	$\varepsilon_3$
$\alpha + \alpha^2$	$\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$
$1 + \alpha^3 + \alpha^5$	$1 + \alpha + \alpha^4 + \alpha^5$
$\alpha^2 + \alpha^3 + \alpha^4$	$1 + \alpha^4 + \alpha^5$
$1 + \alpha^4$	$1 + \alpha$
$1 + \alpha^3 + \alpha^4$	$\alpha + \alpha^3 + \alpha^4$
$1 + \alpha^2 + \alpha^3 + \alpha^5$	$\alpha + \alpha^3 + \alpha^5$
$\alpha + \alpha^3 + \alpha^4$	$1 + \alpha^4 + \alpha^5$
$\alpha^3$	$\alpha^2 + \alpha^4$
$\alpha^2$	$\alpha + \alpha^4 + \alpha^5$

We consider the fourth line,  $\varepsilon_1 = \alpha$ ,  $\varepsilon_2 = 1 + \alpha^4$  and  $\varepsilon_3 = 1 + \alpha$ , in more detail. There are three  $v$ 's giving  $\varepsilon_2$ , namely,

$$1 + \alpha \quad \alpha^4 + \alpha^5 \quad \alpha + \alpha^3 + \alpha^4.$$

Then  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$  iff  $\varepsilon_0$  is any one of these  $v$ 's. And  $\Lambda(Q_{\bar{\varepsilon}}^K) = -1$  iff  $\varepsilon_0 = v + (\alpha/v)^2 = 1 + \alpha^3 + \alpha^5$ .

In total, there are 41,664 quadratic forms on  $K$  with a codimension 2 radical. Of these, 651 have invariant  $-1$ , while 1953 have invariant  $+1$  and the rest have invariant 0.

**Example 2.5.** Let  $K = F_{2^7}$  with primitive element  $\beta$ , a root of  $x^7 + x + 1$ . Suppose  $\varepsilon_1 = 1$ . Again a simple computer search will show there are 21 possible  $\varepsilon_2$ 's, in agreement with by Theorem 2.3. Then  $\varepsilon_3$  can be computed from Theorem 1.5. We list the results, writing  $(i, j)$  for  $\varepsilon_2 = \beta^i$  and  $\varepsilon_3 = \beta^j$ .

$$\begin{array}{cccccc} (21,3) & (37,96) & (41,24) & (42,6) & (47,2) & (55,58) & (59,83) \\ (61,8) & (74,65) & (82,48) & (84,12) & (87,1) & (91,29) & (93,105) \\ (94,4) & (107,64) & (109,78) & (110,116) & (117,32) & (118,39) & (122,16). \end{array}$$

**Example 2.6.** Here we start with an  $\varepsilon_1$  and  $\varepsilon_2$  and then find fields  $K$  that allow this choice. Let  $\varepsilon_1 = 1$  and  $\varepsilon_2 = \gamma$  where  $\gamma^3 = \gamma + 1$ , so that  $\varepsilon_2 \in F_{2^3}$ . Then, as  $x^3 + \gamma x^2 + 1$  is irreducible over  $F_{2^3}$  we have that  $v \in F_{2^9}$ . In fact,  $v = \delta^7$  where  $\delta$  is a root of  $x^9 + x^4 + 1$ . Now  $x^2 + (1/v)x + v$  is irreducible

over  $F_{2^9}$ . Hence  $a, b \in K = F_{2^{18}}$ . The sequence of  $\varepsilon_i$ 's is periodic with period 9:

$$1, \gamma, \gamma, \gamma^2 + 1, \gamma + 1, \gamma, \gamma^2, 1, 0.$$

Hence, if  $\varepsilon_1 = 1$  and  $\varepsilon_2 = \gamma$ ,  $Q_{\bar{\varepsilon}}^K$  has codimension 2 radical iff  $18|k$  and

$$R = \sum_{i=1}^m \varepsilon_i x^{2^i}.$$

Here  $k = 2m$ . Note that the top term  $\varepsilon_m$  is always 0 (recall that  $\varepsilon_m$  is taken modulo  $GF(2^m)$ ).

### 3 Maximal Artin-Schreier curves

The Artin-Schreier curve we consider is:

$$C_{\bar{\varepsilon}} : y^2 + y = xR_{\bar{\varepsilon}}(x).$$

The number of points in  $K$ -projective space on  $C_{\bar{\varepsilon}}$  is:

$$\#C_{\bar{\varepsilon}}(K) = 2^k + \Lambda(Q_{\bar{\varepsilon}}^K)\sqrt{2^{k+w}} + 1,$$

where  $w = \dim \text{rad}(Q_{\bar{\varepsilon}}^K)$ . The Hasse-Weil bound is:

$$\#C_{\bar{\varepsilon}}(K) \leq 2^k + 2^\ell \sqrt{2^k} + 1,$$

where  $2^\ell = \deg R_{\bar{\varepsilon}}$ . Clearly equality will hold in the Hasse-Weil bound only if  $k$  is even.

**Corollary 3.1.** *Suppose  $k = 2m$  and  $\varepsilon_m \in F_{2^m}$ ,  $\varepsilon_{m-1} \neq 0$ . Then the number of points on  $C_{\bar{\varepsilon}}$  equals the Hasse-Weil bound iff  $Q_{\bar{\varepsilon}}^K$  has a radical of codimension 2 and  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$ .*

**Proof:** The conditions on  $\varepsilon_m$  and  $\varepsilon_{m-1}$  yield  $\deg R_{\bar{\varepsilon}} = 2^{m-1}$ , as  $\varepsilon_m$  is taken modulo  $F_{2^m}$ . Now match the two formulas above.  $\square$

If  $\varepsilon_m \notin F_{2^m}$  then  $\deg R_{\bar{\varepsilon}} = 2^m$  and the number of points on  $C_{\bar{\varepsilon}}$  equals the Hasse-Weil bound only if  $\dim \text{rad}(Q_{\bar{\varepsilon}}^K) = k$ , a vacuous case. There are examples of Artin-Schreier curves meeting the Hasse-Weil bound with  $\varepsilon_m \in F_{2^m}$  and  $\varepsilon_{m-1} = 0$ , see [4].

The simplest way to find  $R_{\bar{\varepsilon}}$  satisfying the conditions of Corollary 3.1 is to apply the Construction of Section 2 to  $L := F_{2^m}$ . Namely, choose  $\varepsilon_1 \in L$  and find  $v \in L$  such that  $y^2 + (\varepsilon_1/v)y + v$  splits in  $L$ . Then compute  $\varepsilon_i$  as usual.

**Corollary 3.2.** *The Construction of Section 2 applied to  $L$  yields  $R_{\bar{\varepsilon}}$  with  $\varepsilon_m \in F_{2^m}$ ,  $\varepsilon_{m-1} \neq 0$  and the radical of  $Q_{\bar{\varepsilon}}^K$  having codimension 2. Taking  $\varepsilon_0 = v$  gives  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$  and so the number of points on  $C_{\bar{\varepsilon}}$  equals the Hasse-Weil bound.*

**Proof:** Let  $a, b \in L = F_{2^m}$  be the roots of  $y^2 + (\varepsilon_1/v)y + v$ . Then  $\varepsilon_m = A^{2^m}b \in L$ . If  $\varepsilon_{m-1} = 0$  then  $a^{2^{m-1}}b = ab^{2^{m-1}}$ . Squaring gives  $a^{2^m}b^2 = a^2b^{2^m}$ . As  $a, b \in L$ , we get  $ab^2 = a^2b$  and  $\varepsilon_1 = 0$ , a contradiction. So  $\varepsilon_{m-1} \neq 0$ . The rest follows from Theorem 1.5 and Corollary 3.1.  $\square$

There are other examples of  $R_{\bar{\varepsilon}}$  that satisfy the conditions of Corollary 3.1.

**Example 3.3.** Suppose  $k = 6$ . If  $\varepsilon_3 = a^8b$  is in  $L = F_8$  then  $\varepsilon_3 = \varepsilon_3^8 = ab^8$ . So  $a^8b + ab^8 = 0$ . Now  $a^8b + ab^8$  is the usual formula for  $\varepsilon_3$  (that is, in all cases except  $k = 6$ ) so the formulas of Proposition 1.6 hold. We get  $\varepsilon_2^3 + \varepsilon_1^5 = 0$ . Hence  $\varepsilon_1$  must be a cube.

Conversely, suppose  $\varepsilon_1 = \eta^3$  for some  $\eta \in K$ . Note that if  $\omega$  is a primitive cube root of unity in  $K$  then  $\varepsilon_1 = (\eta\omega)^3$  also. Now let  $\beta$  be a root of  $x^3 + x^2 + 1$ ; note  $\beta \in L$ . Set  $v = \beta\eta^2$ . Then  $\text{tr}_{K/F}(v^3/\varepsilon_1^2) = \text{tr}_{K/F}(\beta^3) = 0$  as  $\beta \in L$ . So  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$ . Following the construction of Section 2, set

$$\varepsilon_2 = \varepsilon_1(v + (\varepsilon_1/v)^2) = \varepsilon_1(\beta\eta^2 + (\eta/\beta)^2) = \eta^5(\beta + 1/\beta^2) = \eta^5.$$

So  $\varepsilon_2^3 + \varepsilon_1^5 = 0$ ,  $a^8b = ab^8$  and  $\varepsilon_3 = a^8b \in L$ .

There are three choices for  $\eta$  and so three choices for  $\varepsilon_2$ . Given  $\varepsilon_1$  and  $\varepsilon_2$ , there are three choices for  $\varepsilon_0$  yielding  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$ , by Theorem 2.3. Hence when  $\varepsilon_1$  is a cube there are exactly nine  $R_{\bar{\varepsilon}}$  satisfying the conditions of Corollary 3.1. When  $\varepsilon_1$  is not a cube there are none. So the number of  $R_{\bar{\varepsilon}}$  with  $\varepsilon_3 \in L$ ,  $\varepsilon_2 \neq 0$ ,  $Q_{\bar{\varepsilon}}^K$  having a codimension 2 radical and  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$  is

$$9 \cdot \frac{2^6 - 1}{3} = 3(2^6 - 1) = 189.$$

**Example 3.4.** Let  $k = 8$ . Suppose  $\varepsilon_4 = a^{16}b \in L = F_{16}$ . As in the previous example, the usual formula for  $\varepsilon_4$ :

$$\varepsilon_4 = \frac{\varepsilon_2}{\varepsilon_1}(\varepsilon_2^6 + \varepsilon_1^5\varepsilon_2^3 + \varepsilon_1^{10})$$

must be 0. So either  $\varepsilon_2 = 0$  or  $\varepsilon_2^3 = \varepsilon_1^5\omega$ , where  $\omega$  is a root of  $x^2 + x + 1$ . Note that  $\omega \in K$  but  $\omega \notin K^{*3}$  as 9 does not divide  $2^8 - 1 = 255$ . Now if



$\varepsilon_2 = 0$  then  $\varepsilon_1 v^3 + \varepsilon_1^3 = 0$  and  $\varepsilon_1^3$  is a cube (equivalently,  $\varepsilon_1$  is a cube, as the order of  $\varepsilon_1$  is odd). If  $\varepsilon_2^3 = \varepsilon_1^5 \omega$  then  $\varepsilon_1^2$  is not a cube, as  $\omega$  is not. We check the converse.

First say  $\varepsilon_1 = \eta^3$  for some  $\eta \in K$ . We follow the Construction of Section 2. Set  $v = \eta^2$ . Then  $y^2 + (\varepsilon_1/v)y + v$  has roots  $\eta\omega, \eta\omega^2$  and so splits in  $K$ . Then  $\varepsilon_2 = \varepsilon_1(v + (\varepsilon_1/v)^2) = 0$ . By Proposition 1.6  $\varepsilon_3 = \varepsilon_1^{11} \neq 0$  and  $\varepsilon_4 = (\eta\omega)^{16}(\eta\omega^2) = \eta^{17} \in F_{16}$  as  $(\eta^{17})^{15} = 1$ .

Next say  $\varepsilon_1^2$  is not a cube. Then  $\varepsilon_1^2$  is in the same coset of  $K^{*3}$  as either  $\omega$  or  $\omega^2$ . We may assume  $\varepsilon_1^2 \omega = \mu^3$ , for some  $\mu \in K$ . Now set  $v = \mu\omega^2$ . Then  $\text{tr}_{K/F}(v^3/\varepsilon_1^2) = \text{tr}_{K/F}(\omega) = 0$  as  $\omega \in F_4$ . So  $y^2 + (\varepsilon_1/v)y + v$  splits in  $K$ . Following the Construction, set  $\varepsilon_2 = \varepsilon_1(v + (\varepsilon_1/v)^2) = \varepsilon_1\mu$ . So  $\varepsilon_2^3 = \varepsilon_1^3\mu^3 = \varepsilon_1^5\omega$ . Hence  $\varepsilon_4 \in F_{16}$ . Lastly,

$$\varepsilon_3 = \frac{1}{\varepsilon_1^2}(\varepsilon_2^3 + \varepsilon_1^5) = \frac{1}{\varepsilon_1^2}(\varepsilon_1^5\omega + \varepsilon_1^5) = \varepsilon_1^2(\omega + 1) = \varepsilon_1^2\omega^2 \neq 0,$$

as desired.

Thus for each cube  $\varepsilon_1$  there is exactly one choice for  $\varepsilon_2$ , namely  $\varepsilon_2 = 0$ . For each non-cube  $\varepsilon_1$  there are three choices for  $\varepsilon_2$ , since there are three choices for  $\mu$ . And, as before, given  $\varepsilon_1, \varepsilon_2$  there are three choices for  $\varepsilon_0$  to get  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$ . Hence the number of  $R_{\bar{\varepsilon}}$  with  $\varepsilon_4 \in F_{16}, \varepsilon_3 \neq 0, Q_{\bar{\varepsilon}}^K$  having a radical of codimension 2 and  $\Lambda(Q_{\bar{\varepsilon}}^K) = +1$  is:

$$3 \cdot \frac{2^8 - 1}{3} + 9 \cdot \frac{2(2^8 - 1)}{3} = 7(2^8 - 1) = 1785.$$

These then are the maximal Artin-Schreier curves over  $K = F_{2^8}$ .

Additional computations suggest that each case behaves like one of the two examples above and that the number of  $R_{\bar{\varepsilon}}$  that satisfy the conditions of Corollary 3.1 is:

$$\begin{cases} 3(q-1), & \text{if } m \text{ is odd} \\ 7(q-1), & \text{if } m \text{ is even.} \end{cases}$$

But we are unable to show this.

Lastly, [9] deserves a comment since it has results that appear similar to ours. There is, in fact, little overlap. Equation (2) here is a special case ( $w = m - 2$ ) of Equation (7) in [9]. But the concerns are different. In [9] they seek only to determine the degree of  $R(x)$  while we seek to determine the coefficients. When the codimension 2 radical case is considered in [9], they

obtain either no information on the coefficients (Section 4, I) or information only on the top coefficient (Section 5, I).

[9] also constructs curves that attain the Hasse-Weil bound but they are fibre products of Artin-Schreier curves, rather than the single curves considered here. Taking  $r = 1$  in [9] Proposition 5.2 (ii) does prove the existence of one Artin-Schreier curve attaining the Hasse-Weil bound. In this section, we found many maximal Artin-Schreier curves (Corollary 3.2) and have suggested a method to find all of them.

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